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Discrete Optimization

# The equitable location problem on the plane

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**Abstract**

This paper considers the problem of locating $M$ facilities on the unit square so as to minimize the maximal demand faced by each facility subject to closest assignments and coverage constraints. Focusing on uniform demand over the unit square, we develop upper and lower bounds on feasibility of the problem for a given number of facilities and coverage radius. Based on these bounds and numerical experiments we suggest a heuristic to solve the problem. Our computational results show that the heuristic is very efficient, as the average gap between its solutions and the lower bound is 4.34%.


*Keywords:* Location; Heuristics; Facilities planning and design

## 1. Introduction

Consider the problem of locating $M$ identical facilities on a certain geographical region with spatially distributed demand. The customers will patronize the closest facility and the capacity of each facility must be sufficient to meet all expected demand. Assuming the capacity is costly and that all customers must be covered, the problem of selecting the locations to minimize the total cost is equivalent to finding locations that minimize the demand at the *busiest* facility. This is because the capacity of the facilities will be determined by the busiest facility—any facility seeing less demand than the busiest one will have some unused capacity (in fact, the ideal locations would be such that the demand at all facilities is identical).

This paper considers the problem of locating $M$ facilities on the unit square so as to minimize the maximal demand faced by any facility subject to closest assignments and coverage constraints. By minimizing the demand faced by the facilities, we are in fact reducing the difference in demand rates between the busiest

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and the least busy facilities—hence we refer to this problem as the equitable location problem (ELP). When $M$ is not given, we show how to find the $M$ for which ELP is feasible.

As an example consider the problem of locating cellular phone towers in a given region. To reduce the procurement and maintenance costs of these towers they are identical in capacity. We wish to cover the demand in this region using $M$ identical towers, each tower can cover an area of a radius $r$ where demand is allocated to the closest tower. Because towers' capacity is correlated with the demand they cover, this demand dictates their cost. Thus, to minimize costs we should minimize the maximal demand faced by a tower.

The original motivation for investigating the ELP (which is a deterministic problem) is it being a major component in the stochastic capacity and facility location problem (SCFLP) discussed in Baron et al. (2005) (henceforth denoted as BBK). The SCFLP focuses on three sources of uncertainty: the timing, location, and actual amount of the demand generated by the customers. This problem optimizes three types of decision variables: (i) the number of facilities to be located; (ii) the location of the facilities; (iii) the service capacity of each facility. Readers interested in stochastic location problems are referred to Berman and Krass (2002).

The approach taken in BBK to solve the SCFLP is based on the following. For a given $M$, the ELP is solved to provide optimal location for the facilities and maximal demand rate to a facility, $\lambda_{\max}$. While BBK developed efficient methods to search over $M$ and estimate the required capacity for any space, they solved ELP on a line. In this paper, we investigate the solution of ELP on the unit square, with a uniform demand (with the exception of Section 3.2.1, discussing more general demand structure).

We strongly believe that ELP is an important problem in its own right. To the best of our knowledge, the only paper that have some similarity to the ELP is Berman et al. (2005) who studied the problem of minimizing the maximum total weight faced by facilities on a network.

To capture the closest assignment constraints, we reformulate ELP using Voronoi diagrams. We then suggest three methods to generate initial solutions for ELP and three procedures to improve these initial solutions. Finally, based on our numerical experiments, we develop a heuristic for solving ELP. We focus on the unit square with a uniform demand. However, in view of the vast literature on using Voronoi diagrams (e.g., chapter 10 and 12 of Gersho and Gray (1991), Aurenhammer (1991), Suzuki and Okabe (1995), and references therein) we believe that the approach presented should work in more general settings.

In Section 2 we present ELP faced by the decision maker over a plane and cite relevant results from BBK. Section 3, reformulates ELP using Voronoi diagrams and discusses some feasibility issues of this problem including finding the minimal $M$ for which ELP is feasible. Next, Section 4 develops a few heuristics for solving ELP, tests their performances, and recommend a heuristic for solving ELP. Finally, we briefly discuss how to extend the approach developed in this paper to different demand settings and spaces.

## 2. The ELP

Let $P$ be a space $P \subset R^2$ equipped with some norm $\|\cdot\|$ and $M > 0$ denote the number of facilities. Let $\mathbf{x}_j$ be a vector denoting the location of facility $j$, $\mathbf{x}_j \in P$, let $I^j_{\mathbf{x}} = 1$ if the $j$th facility is the closest one to $\mathbf{x}$ and $I^j_{\mathbf{x}} = 0$ otherwise, and let $R(\mathbf{x}_j) \equiv max_{\mathbf{x}\in P}\|\mathbf{x}, \mathbf{x}_j\| I^j_{\mathbf{x}}$ be the maximum travel distance of customers' assigned to facility $j$. Assume that at each $\mathbf{x} \in P$ customers' demand rate is $\lambda(\mathbf{x})$ such that $\int_{\mathbf{x}\in P} \lambda(\mathbf{x})\mathrm{d}\mathbf{x} = \Lambda < \infty$, thus $\lambda_{\mathbf{x}_j} = \int_{\mathbf{x}\in P} I^j_{\mathbf{x}}\lambda(\mathbf{x})\mathrm{d}\mathbf{x}$ is the arrival rate to the $j$th facility. Let $r$ be an exogenously given distance, which is the maximum distance allowed from a customer to a facility, and let $\varepsilon$ be the smallest distance allowed between distinct facilities.

Consider the following optimization problem called the "equitable location problem" (ELP($M$)):

$$
\begin{aligned}
\min \quad & \lambda^{\max} && (1)\\
\text{S.t.} \quad & \|\mathbf{x}_i, \mathbf{x}_j\| \geqslant \varepsilon \quad \forall i, j = 1, \ldots, M,\ i \neq j, \\
& \lambda^{\max} \geqslant \lambda_{\mathbf{x}_j} \quad \forall j = 1, \ldots, M, \\
& R(\mathbf{x}_j) \leqslant r \quad \forall j = 1, \ldots, M, \\
& \mathbf{x}_j \in P \quad \forall j = 1, \ldots, M,
\end{aligned}
$$

where $\lambda_{\mathbf{x}_j}$ is the demand rate to the $j$th facility. As mentioned earlier, the objective of ELP($M$) is to locate $M$ distinct (first set of constraints) facilities such that the coverage (third set of constraints) and closest assignment constraints are satisfied and the demand faced by the "busiest" facility is as small as possible. (In the second set of constraints, when we calculate $\lambda_{\mathbf{x}_j}$ we make sure that closest assignments are maintained.)

The following definition was introduced in BBK:

**Definition 1** (*Equitable Facility Configuration*). We say that a location vector **x** represents an Equitable Facility Configuration (EFC) if the demand rates to all facilities are the same, i.e., if $\lambda^{\max} = \Lambda/M$, where $\Lambda$ is the total demand rate.

It is clear that if a feasible EFC location vector exists in the ELP($M$) model, then this location vector is optimal. In this paper, we focus on solving the ELP($M$) problem where $P$ is the unit square with uniform demand and the $L^2$ distance norm, defined by:

$$\|\mathbf{x}_i, \mathbf{x}_j\| = \sqrt{\|x_i - x_j\|^2 + \|y_i - y_j\|^2}.$$

## 3. Reformulating the ELP(M) problem

Here we formulate ELP and discuss its feasibility.

### 3.1. Formulation using Voronoi diagram

Given $M$ facilities, the closest assignment constraints with a given distance norm, divides the plane to $M$ areas using a Voronoi diagram of this distance norm. A comprehensive survey on Voronoi diagrams is given in Aurenhammer (1991) and Okabe et al. (2000), where a wide range of applications of Voronoi diagrams are discussed, a survey on the application of Voronoi diagrams in facility location problems is given in Suzuki and Okabe (1995).

We denote by $V_i$ the Voronoi region associated with the $i$th facility. Then, the demand rate to this facility is:

$$\lambda_{\mathbf{x}_i} = \frac{1}{\Lambda} \int_{\mathbf{x} \in V_i} \mathrm{d}x\mathrm{d}y \quad \forall i = 1, \ldots, M.$$

Assuming that the Voronoi diagram is given, we rewrite ELP($M$) in a more convenient form:

$$\begin{aligned} \min \quad & \lambda^{\max} \\ \text{S.t.} \quad & \lambda_{\mathbf{x}_i} \leqslant \lambda^{\max} \quad \forall i = 1, \ldots, M, \\ & \|\mathbf{x}_i, \mathbf{x}_j\| \geqslant \varepsilon \quad \forall i, j = 1, \ldots, M, i \neq j, \\ & \|\mathbf{x}_i, \mathbf{x}\| \leqslant r \quad \forall i = 1, \ldots, M, \mathbf{x} \in V_i. \end{aligned} \tag{2}$$

We found this version of the ELP($M$) problem more tractable than the original one, thus the rest of the paper is focused on solving it.

### 3.2. Feasibility issues

When the value of $M$ and $r$ are small, ELP($M$) may be infeasible and therefore no feasible EFC exists. In this section, we first give sufficient conditions for the existence of EFC, given the closest assignment constraints but ignoring the coverage constraints. We then present a lower bound on the number of facilities required to satisfy the coverage constraints.

#### *3.2.1. No coverage constraints ($r \geqslant \sqrt{2}$) – existence of EFC*

When $r \geqslant \sqrt{2}$, all points on the unit square are covered from each facility within the square, thus the coverage constraint can be ignored. In such cases, when demand is uniform over $P$, an EFC can always be found by locating the $M$ facilities evenly on the line of $(0, 0.50)$ to $(1, 0.5)$ (or $(0.5, 0)$ to $(0.5, 1)$). Furthermore, for a general demand process $\lambda(x, y)$ we show that if $\lambda(x, y)$ increases with $x$ or $y$, decreases with $x$ or $y$, or is bell-shaped with center at point $(0.5, 0.5)$ then an EFC exists for any given number $M$. These conclusions follow the next result from BBK for the case that the space considered is a line segment. Define $\lambda(x)$ as the demand rate at $x \in [0, 1]$, then $\Lambda = \int_0^1 \lambda(x)\mathrm{d}x$ and:

**Proposition 1.** *Suppose $\lambda(x)$ is non-increasing or non-decreasing on $[0, 1]$ or non-decreasing on $[0, z]$ and non-increasing on $[z, 1]$ for some $z \in [0, 1]$, then an EFC exists.*

When $\lambda(x, y)$ satisfies one of the conditions above, we can aggregate the demand from each point on the square to the closest point on the line $(0, 0.50)$ to $(1, 0.5)$ (or $(0.5, 0)$ to $(0.5, 1)$). This demand aggregation transforms the ELP on a plane into an ELP on a line with $\lambda(x)$ satisfying the condition in Proposition 1. Thus, an EFC exists.

#### *3.2.2. Active coverage constraints ($r < \sqrt{2}$) – feasibility of ELP(M)*

It is easy to see that if ELP($M_0$) is feasible, then ELP($M$) is feasible for any $M > M_0$. Yet, if an EFC exists for $M_0$, we have no guarantee that an EFC exists for each $M > M_0$. Ensuring that the coverage constraints hold for every point on $P$ is equivalent to covering $P$ by $M$ identical circles with radius $r$. Thus, because $P$ is a unit square, $M$ must satisfy

$$M\pi r^2 \geqslant 1 \Rightarrow M_{\mathrm{lb}} = \left\lceil \frac{1}{\pi r^2} \right\rceil,$$

where $M_{\mathrm{lb}}$ is the lower bound on $M$. This lower bound can be improved by the "honeycomb conjecture", proved in Hales (2001). This result states that "any partition of the plane into regions of equal area has perimeter at least that of the regular hexagonal honeycomb tiling". Thus, dividing the plane into regions with a maximal radius $r$ can be done with hexagons. The area of a hexagon is $3\sqrt{3}r^2/2$, therefore, $M_{\mathrm{lb}}$ can be improved to:

$$M_{\mathrm{lb}} 3\sqrt{3}r^2/2 \geqslant 1 \Rightarrow M_{\mathrm{lb}} = \left\lceil \frac{2}{3\sqrt{3}r^2} \right\rceil.$$

In practice we find the minimal $M$ for a given $r$ by solving a sequence of *p-center* problem (see Suzuki and Drezner (1996)). Initially, let $p = M_{\mathrm{lb}}$, solve the corresponding *p-center* problem, if the resulting maximum radius is less than or equal to $r$, stop, ELP($M$) is feasible for any $M \geqslant p$; otherwise, increase the value of $p$ by 1 and continue the procedure until the resulting maximum radius is less than or equal to $r$. A more effective search would be a bisection search that also requires an upper bound on $M$. We discuss such bounds in the next subsection.

### *3.3. Upper bounds*

Here we develop upper bounds on the value of $M$, (these bounds can also be used to develop an upper bound on $\lambda_{\max}$ and consequently on the cost of the SCFLP problem analyzed in Baron et al. (2005)). The first upper bound on $M$ locates $M_1$ facilities similarly to the approach used in Suzuki and Drezner (1996) to solve the $M$-center problem. The second one is to locate $M_2$ facilities where an EFC is easy to reach for $M_2$.

*3.3.1. Upper bound 1*

The first upper bound is calculated by dividing region $P$ into rectangles so that each rectangle can be covered by a circle of radius $r$ as shown in Fig. 1.

Let $h_1 = \lceil \frac{1}{\sqrt{2}r} \rceil$, $a_1 = \frac{1}{2h_1}$, $b_1 = \sqrt{r^2 - a_1^2}$, $n_1 = \lfloor \frac{1}{2b_1} \rfloor$, $b_2 = \frac{1-2n_1b_1}{2}$, $a_2' = \sqrt{r^2 - b_2^2}$, $h_2 = \lceil \frac{1}{2a_2'} \rceil$, $a_2 = \frac{1}{2h_2}$. Then rectangle $A$ is a $2a_1$ by $2b_1$ rectangle and rectangle $B$ is a $2a_2$ by $2b_2$ rectangle. For example, if $r = 0.27$, $h_1 = 3$, $a_1 = 1/6$, $b_1 = 0.2124$, $n_1 = 2$, $b_2 = 0.0752$, $a_2' = 0.2593$, $h_2 = 2$ and $a_2 = 0.25$.

Given $r$, a feasible solution for ELP can be obtained by locating

$$M_1 = n_1h_1 + h_2,$$

(i.e., $n_1h_1$ $A$ rectangles and $h_2$ $B$ rectangles) facilities at the center of the $M_1$ rectangles. The coordinates of the facilities are

$$x_i = \begin{cases} (2\mathrm{mod}(i-1, h_1) + 1)a_1 & \text{if } i \leqslant n_1h_1 \\ (2(i - n_1h_1) - 1)a_2 & \text{otherwise} \end{cases}$$

$$y_i = \begin{cases} (2\lfloor (i-1)/h_1 \rfloor + 1)b_1 & \text{if } i \leqslant n_1h_1 \\ 2n_1b_1 + b_2 & \text{otherwise.} \end{cases}$$

Note that each facility may not exactly cover the customers in their corresponding rectangle due to the closest assignment constraints. However, the customer can always find a facility within $r$ units of distance.

Draw the Voronoi diagram for the $M_1$ facilities. For facility $i$, the demand rate $\lambda_i$ can be obtained by calculating the area $P_i$ of each Voronoi region and $\lambda^{\max} = \max\{\lambda_i\colon i = 1, \ldots, M_1\}$.

*3.3.2. Upper bound 2*

The second upper bound is calculated by dividing region $P$ into squares so that each square can be covered by a circle of radius $r$.

Let $n_0 = \lceil 1/(\sqrt{2}r) \rceil$ and $M_2 = n_0^2$. If we divide $P$ into $n_0$ by $n_0$ squares, then each square has a side of $a = 1/n_0$. If we locate a facility at the center of each square, then the customers in each square are served by the corresponding facility and an EFC is reached with $\lambda_{\max}(M) = a^2 = 1/M_2$.

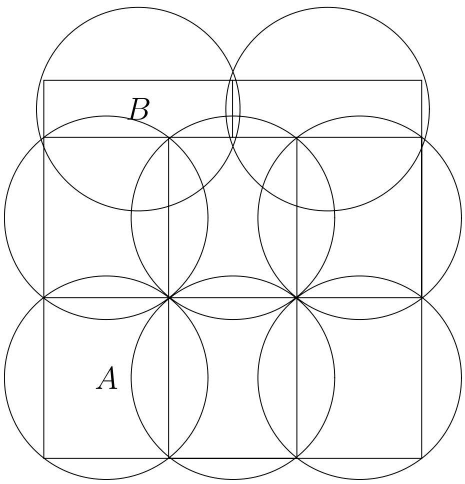



Fig. 1. Divide $P$ into rectangles: $r = 0.27$.

## 4. Heuristics for the ELP(M) problem

In this section, we develop a heuristic to find an approximation of $\lambda_{\max}(M)$. We present three sets of initial locations for the $M$ facilities and three heuristics to improve facilities' locations. Based on our numerical experiments we then suggest a heuristic to solve the ELP($M$) problem.

### 4.1. Initial facility locations

We consider three methods to generate an initial set of facility locations. The first is to randomly generate $M$ points in the unit square. The second is to use the algorithm developed by Suzuki and Drezner (1996) to solve the $M$-center problem on a square. We call the third the Rectangle Center and present it below.

For any given $M$, let $n = \lceil \sqrt{M} \rceil$. We divide $P$ into $M$ subregions as follows:

- If $M = n^2$, then we divide $P$ into $M$ squares, each with a size of $1/n$ by $1/n$.
- If $M = n(n-1)$, then we divide $P$ into $M$ rectangles, each with a size of $1/n$ by $1/(n-1)$.
- If $(n-1)^2 < M < n(n-1)$, then we divide $P$ into $(n-1)$ strips. Among the $(n-1)$ strips, the first $n_1 = n(n-1) - M$ strips have $n-1$ rectangles with a size of $a_1 = 1/(n-1)$ by $b_1 = (n-1)/M$, the other $n_2 = n - 1 - n_1$ strips have $n$ rectangles with a size of $a_2 = 1/n$ by $b_2 = n/M$.
- If $n(n-1) < M < n^2$, then we divide $P$ into $n$ strips. Among the $n$ strips, the first $n_1 = n^2 - M$ strips have $n-1$ rectangles with a size of $a_1 = 1/(n-1)$ by $b_1 = (n-1)/M$, the other $n_2 = n - n_1$ strips have $n$ rectangles with a size of $a_2 = 1/n$ by $b_2 = n/M$.

For example, when $M = 10$, $n = 4$ and $(n-1)^2 < M < n(n-1)$. $P$ can be divided into $n - 1 = 3$ strips. $n_1 = n(n-1) - M = 2$ strips have $n - 1 = 3$ rectangles with a size of $a_1 = 1/(n-1) = 1/3$ by $b_1 = (n-1)/M = 3/10$, the other $n_2 = n - 1 - n_1 = 1$ strips have $n = 4$ rectangles with a size of $a_2 = 1/4$ by $b_2 = n/M = 4/10$. The 10 rectangles are illustrated in Fig. 2. The size of rectangle $A$ is 1/3 by 3/10 and the size of rectangle $B$ is 1/4 by 4/10.

When $M = 15$, $n = 4$ and $n(n-1) < M < n^2$. $P$ can be divided into $n = 4$ strips. $n_1 = n^2 - M = 1$ strips have $n - 1 = 3$ rectangles with a size of $a_1 = 1/(n-1) = 1/3$ by $b_1 = (n-1)/M = 1/5$, the other $n_2 = n - n_1 = 3$ strips have $n = 4$ rectangles with a size of $a_2 = 1/4$ by $b_2 = n/M = 4/15$. The 15 rectangles are illustrated in Fig. 3. The size of rectangle $A$ is 1/3 by 1/5 and the size of rectangle $B$ is 1/4 by 4/15.

If we locate a facility at the center of each rectangle, the coordinates of the facilities are:

$$x_i = \begin{cases} (\mathrm{mod}(i-1, n-1) + 0.5)a_1 & \text{if } i \leqslant n_1(n-1) \\ (\mathrm{mod}(i - n_1(n-1) - 1, n) + 0.5)a_2 & \text{otherwise} \end{cases}$$

$$y_i = \begin{cases} (\lfloor (i-1)/(n-1) \rfloor + 0.5)b_1 & \text{if } i \leqslant n_1(n-1) \\ (n_1 b_1 + \lfloor (i - n_1(n-1) - 1)/n \rfloor + 0.5)b_2 & \text{otherwise.} \end{cases}$$

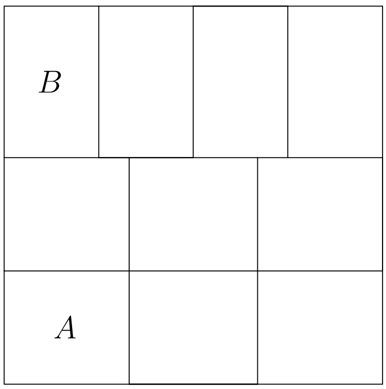



Fig. 2. Divide $P$ into 10 subregions with equal area.

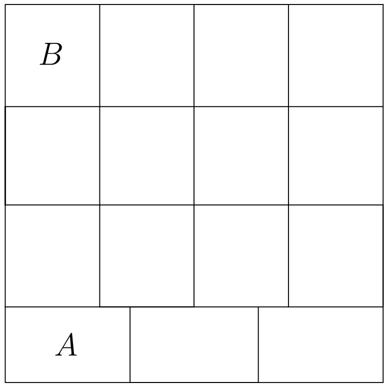



Fig. 3. Divide $P$ into 15 subregions with equal area.

Observe that for a given $r$ and $M$, the Rectangle Center method described above may fail to find a feasible solution, i.e., some of the points in $P$ may not be covered by any facility due to the coverage constraints. We have

**Proposition 2.** *For any* $m \geqslant 3$, $r = \sqrt{2}/(2m)$, $M = m^2 + 1$, *the highest row of rectangles cannot be covered by the facility located at its center.*

**Proof.** In this case, $n = m + 1$, $a_2 = 1/n$, $b_2 = n/M$.

$$a_2^2 + b_2^2 - 4r^2 = \frac{1}{(m+1)^2} + \frac{(m+1)^2}{M^2} - \frac{2}{m^2} = \frac{1}{(m+1)^2} + \frac{(m+1)^2}{(m^2+1)^2} - \frac{2}{m^2} = \frac{2m^4 + 4m^3 - 4m^2 - 4m - 2}{m^2(m+1)^2(m^2+1)^2} > 0. \quad \square$$

However, if we move the facilities in these rectangle a little higher, the upper uncovered point may be covered by the new location and the lower uncovered point may be covered by the facilities located at the center of the rectangles on the $(m - 1)$th strip which have a smaller height.

We can also prove the following proposition.

**Proposition 3.** *A feasible solution always exists when* $m = \lceil 1/(\sqrt{2}r) \rceil$, $M > m^2 + 1$.

**Proof.** Since $m = \lceil 1/(\sqrt{2}r) \rceil$, $4r^2 \geqslant 2/m^2$.

$n = \lceil \sqrt{M} \rceil \geqslant m + 1$, $a_1 = 1/(n-1) \leqslant 1/m$, $b_1 = (n-1)/M < 1/(n-1) \leqslant 1/m$, $a_2 = 1/n < 1/m$, $b_2 = n/M$. We have

$$a_1^2 + b_1^2 - 4r^2 < 1/m^2 + 1/m^2 - 2/m^2 = 0.$$

When $n = m + 1$, i.e., $m^2 + 1 < M < (m+1)^2$ let $M = m^2 + a$, we have

$$a_2^2 + b_2^2 - 4r^2 \leqslant \frac{1}{(m+1)^2} + \frac{(m+1)^2}{M^2} - \frac{2}{m^2} = \frac{1}{(m+1)^2} + \frac{(m+1)^2}{(m^2+a)^2} - \frac{2}{m^2}$$
$$= -\frac{(2a-4)m^4 + (8a-4)m^3 + (a^2+4a-1)m^2 + 4a^2m + 2a^2}{m^2(m+1)^2(m^2+a)^2}.$$

Since $a > 1$, $a_2^2 + b_2^2 - 4r^2 < 0$.

When $n > m + 1$, $M > (n-1)^2 > mn$, $a_2 = 1/n < 1/m$, $b_2 = n/M < 1/m$:

$$a_2^2 + b_2^2 - 4r^2 < 1/m^2 + 1/m^2 - 2/m^2 = 0. \quad \square$$

### 4.2. Procedures to find the maximal arrival rate (given M)

When the locations of the $M$ facilities are given, the demand rate for each facility can be calculated by finding the Voronoi diagram and calculating the area $P_i$ of each Voronoi region and $\lambda_{\max}(M)$ is the maximal area. Initial solutions can be based on either one of the methods discussed in 4.1. Here, we present three procedures to improve the locations given by an initial solution.

#### 4.2.1. Procedure 1 for ELP(M)

This procedure starts with the set of initial solutions generated by the rectangle center method described above. Let $n = \lceil\sqrt{M}\rceil$, there are either $(n-1)$ or $n$ rows of facilities. In either case, the lower rows have $(n-1)$ facilities and the upper rows have $n$ facilities. Let row $i$ (counted from the bottom) be the highest row that has $(n-1)$ facilities, i.e., row $(i+1)$ has $n$ facilities. For example, when $M = 10$, $i = 2$ and when $M = 15$, $i = 1$.

It is easy to see that when the facilities are located at this initial set, the facilities on row $i$ will have a demand rate greater than $1/M$. If we move the $n$ facilities on row $(i+1)$ slightly towards these $(n-1)$ facilities (i.e., a little lower in the square), the value of $\lambda_{\max}(M)$ will decrease. We continue this procedure until there is no improvement.

#### 4.2.2. Procedure 2 for ELP(M)

This procedure is based on local adjustments. It starts with any set of initial solutions described in Section 4.1. Then it finds the one with the largest (smallest) demand rate. Move the facilities around the neighborhood to decrease (increase) the largest (smallest) demand rate.

Let $\epsilon$ be a given small positive number, $P_i$ be the Voronoi region of facility $i$, $\lambda_i$ be the demand rate of facility $i$, $i_{\min} = \arg\min\{\lambda_i\text{: } i = 1,\ldots,M\}$, $i_{\max} = \arg\max\{\lambda_i\text{: } i = 1,\ldots,M\}$, $\lambda_{\min} = \lambda_{i_{\min}}$, $\lambda_{\max} = \lambda_{i_{\max}}$, and $N_i = \{j\text{: } P_j$ shares at least an edge with $P_i\}$. We assume for the time being that $\lambda_{i_{\min}}$ and $\lambda_{i_{\max}}$ are unique.

*Step 1*. (Initialization) For a given $M$, find the initial facility location by one of the methods listed in 4.1. Denote $\lambda_{\max}(M) = \infty$. Make obvious improvements.

*Step 2*. Find the Voronoi regions for the facilities and their demand rates $\lambda_i$. Find $i_{\min}$, $i_{\max}$, $\lambda_{\min}$ and $\lambda_{\max}$.
If $\lambda_{\max} < \lambda_{\max}(M)$, update $\lambda_{\max}(M) = \lambda_{\max}$.
If $(\lambda_{\max} - \lambda_{\min}) < \epsilon$ or the change of $(\lambda_{\max} - \lambda_{\min})$ is negligible, stop; otherwise, go to *Step 3*.

*Step 3*. (Relocation) Find a facility to relocate so that either $\lambda_{\max}$ will decrease or $\lambda_{\min}$ will be increased.
- *3.1.* If $\lambda_{\max} - 1/M > 1/M - \lambda_{\min}$, consider facility $i_{\max}$ and its neighborhood $N_{i_{\max}}$. Let $j$ be the facility with the smallest demand rate in $N_{i_{\max}}$. If $i_{\max}$ is unique, move $j$ toward $i_{\max}$, this may reduce the demand rate for $i_{\max}$ and increase the demand rate for $j$. Go to *Step 2*.
- *3.2.* Else, consider facility $i_{\min}$ and its neighborhood $N_{i_{\min}}$. Let $j$ be the facility with the largest demand rate in $N_{i_{\min}}$. Move $i_{\min}$ toward $j$, this may increase the demand rate for $i_{\min}$ and decrease the demand rate for $j$. Go to *Step 2*.

#### 4.2.3. Procedure 3 for ELP(M)

This procedure is based on global adjustments. It starts with any set of initial solutions described in Section 4.1. Then it finds the facilities with the largest and smallest demand rate. It moves a facility with the largest demand rate toward one with the smallest demand rate to decrease the gap.

*Step 1*. (Initialization) For a given $M$, find the initial facility location by one of the methods listed in 4.1. Denote $\lambda_{\max}(M) = \infty$.

*Step 2*. Same as *Step 2* in Procedure 2.

*Step 3*. (Relocation) Move $i_{\min}$ toward $i_{\max}$, this may reduce the demand rate for $i_{\max}$ and increase the demand rate for $i_{\min}$. Go to *Step 2*.

Table 1
Movement of facilities

| Iter | 1 | 2 | 3 | 4 | 5 | 6 | $\lambda_{max}$ | $\lambda_{min}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | (0.1, 0.4) | (0.2, 0.9) | (0.3, 0.6) | (0.3, 0.7) | (0.1, 0.5) | (0.2, 0.6) | 0.38625 | 0.03125 |
| 2 | (0.25, 0.167) | (0.25, 0.833) | (0.75, 0.167) | (0.75, 0.833) | (0.1, 0.5) | (0.2, 0.6) | 0.247165 | 0.0675 |
| 3 | (0.25, 0.167) | (0.25, 0.833) | (0.75, 0.167) | (0.75, 0.833) | (0.1, 0.5) | (0.45, 0.4) | 0.225076 | 0.103445 |
| 4 | (0.25, 0.167) | (0.25, 0.833) | (0.75, 0.167) | (0.75, 0.833) | (0.1, 0.5) | (0.5, 0.5) | 0.2045 | 0.10725 |
| 5 | (0.25, 0.167) | (0.25, 0.833) | (0.75, 0.167) | (0.75, 0.833) | (0.3, 0.5) | (0.5, 0.5) | 0.2045 | 0.1355 |
| 6 | (0.25, 0.167) | (0.25, 0.833) | (0.75, 0.167) | (0.75, 0.833) | (0.3, 0.5) | (0.7, 0.5) | 0.16875 | 0.1625 |

#### 4.2.4. Discussion and an example

Procedures 2 and 3 are more difficult to implement than Procedure 1 since controlling facilities' movement requires more judgment in choosing the direction and the distance of shift. Moreover, when there are coverage constraints, the step size of the movement must keep the feasibility of the solution. This can be done by reducing the step size once the last step size leads to an infeasible solution. In the examples to follow, we discuss our choices for the direction and step sizes.

We presented the procedures above assuming that the facilities recognized in the search steps are uniquely determined. In the case when these facilities are not unique the movement should be directed towards (or from) the average location of these multiple facilities. *Step 4* in the following example demonstrate such a case and its solution.

**Example 1.** Consider the case when $M = 6$, $r = 1$, and $\epsilon = 0.01$. The Voronoi diagrams associated with location of facilities in the following steps is given in Appendix A, the locations, $\lambda_{max}$ and $\lambda_{min}$ are given in Table 1.

1. Initially, the facilities are located at points (0.1, 0.4), (0.2, 0.9), (0.3, 0.6), (0.3, 0.7), (0.1, 0.5), and (0.2, 0.6) (Diagram 1). Facility 3 has the maximum demand rate of 0.38625 and facility 5 has the minimum demand rate of 0.03125. As obvious improvements, we observe that facilities 1, 2, 3 and 4 cover the 4 corners of the unit square, thus we move them closer to the corners (Diagram 2). After this, Facility 3 still has the maximum demand rate of 0.247165 and facility 5 has the minimum demand rate of 0.0675.
2. In Diagram 2, facility 6 has the smallest demand rate in the neighborhood of facility 3. After moving facility 6 toward facility 3 (see Diagram 3), facility 2 has the maximum demand rate of 0.225076 and facility 5 has the minimum demand rate of 0.103445.
3. In Diagram 3, facility 6 has the smallest demand rate in the neighborhood of facility 2. After moving facility 6 toward facility 2 (see Diagram 4), facilities 2 and 3 have the maximum demand rate of 0.2045 and facility 5 has the minimum demand rate of 0.10725.
4. Since 0.2045 is close to 1/6 (the EFC demand rate), we consider the facility with the minimum demand rate – facility 5. Since the facilities with the maximum demand rate are all located on the right side of facility 5, we move facility 5 to the right (see Diagram 5). After this, facilities 2 and 3 have the maximum demand rate of 0.2045 and facility 6 has the minimum demand rate of 0.1355.
5. Since the facilities with the maximum demand rate are all located on the right side of facility 6, we move facility 6 to the right. After this, facilities 1, 2, 3 and 4 have the maximum demand rate of 0.16875 and facilities 5 and 6 have the minimum demand rate of 0.1625. Since $0.16875 - 0.1625 \leqslant \epsilon$, Stop.

### 4.3. Performance of the three procedures

In this section, we test the performance of the procedures started with different initial solution sets, assuming that the coverage constraints are inactive. First, we briefly describe how we make the location adjustment in our computational experiments.

Table 2
Quality of procedures for ELP(*M*) (in percentage)

| *M* | Ideal EFC | Percent gap from ideal $H(0,2)$ | $H(0,3)$ | $H(1,3)$ | $H(2,1)$ | $H(2,2)$ | $H(2,3)$ | $H(3,1)$ | $H(3,2)$ | $H(3,3)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.33333 | 12.500 | 6.250 | 0.001 | 7.169 | 13.261 | 0.001 | 10.901 | 6.148 | 2.353 |
| 5 | 0.20000 | 13.596 | 9.606 | 1.111 | 44.114 | 13.596 | 9.606 | 44.179 | 4.101 | 9.606 |
| 7 | 0.14286 | 7.713 | 6.828 | 2.061 | 65.375 | 7.713 | 6.828 | 36.182 | 7.713 | 6.828 |
| 8 | 0.12500 | 2.381 | 5.093 | 1.974 | 58.156 | 2.347 | 5.093 | 61.003 | 2.381 | 5.093 |
| 10 | 0.10000 | 14.013 | 6.680 | 0.666 | 80.821 | 10.378 | 6.680 | 183.810 | 13.893 | 6.680 |
| 11 | 0.09091 | 7.324 | 6.333 | 2.726 | 80.263 | 7.324 | 6.333 | 93.635 | 7.324 | 6.333 |
| 13 | 0.07692 | 14.520 | 5.539 | 1.088 | 34.575 | 10.136 | 5.539 | 74.521 | 14.520 | 5.539 |
| 14 | 0.07143 | 5.149 | 5.093 | 2.176 | 127.377 | 4.956 | 5.093 | 253.543 | 5.149 | 5.093 |
| 15 | 0.06667 | 2.744 | 4.613 | 1.817 | 122.374 | 2.744 | 4.613 | 146.665 | 2.744 | 4.613 |
| 17 | 0.05882 | 9.760 | 5.146 | 0.687 | 79.923 | 9.756 | 4.742 | 159.352 | 9.760 | 5.146 |
| 18 | 0.05556 | 5.916 | 5.013 | 2.259 | 122.392 | 5.916 | 4.558 | 155.382 | 5.916 | 5.013 |
| 19 | 0.05263 | 5.424 | 4.871 | 2.194 | 163.044 | 4.723 | 4.491 | 128.253 | 5.424 | 4.871 |
| 21 | 0.04762 | 11.382 | 4.566 | 1.040 | 54.967 | 11.246 | 4.384 | 122.755 | 11.382 | 4.566 |
| 22 | 0.04545 | 4.842 | 4.401 | 1.974 | 139.070 | 4.842 | 4.401 | 78.656 | 4.842 | 4.401 |
| 23 | 0.04348 | 4.521 | 4.230 | 1.892 | 86.777 | 4.521 | 4.230 | 86.590 | 4.521 | 4.230 |
| 24 | 0.04167 | 3.013 | 4.050 | 1.807 | 116.881 | 3.013 | 4.050 | 202.633 | 3.013 | 4.050 |
| 26 | 0.03846 | 9.589 | 4.203 | 0.580 | 97.606 | 8.755 | 4.203 | 80.699 | 9.589 | 4.203 |
| 27 | 0.03704 | 5.134 | 4.141 | 1.894 | 79.525 | 5.019 | 4.077 | 293.277 | 5.134 | 4.141 |
| 28 | 0.03571 | 5.029 | 4.076 | 1.862 | 99.648 | 4.912 | 3.823 | 135.789 | 5.029 | 4.076 |
| 29 | 0.03448 | 4.363 | 4.009 | 1.830 | 152.377 | 4.067 | 3.797 | 99.716 | 4.363 | 4.009 |
| 31 | 0.03226 | 11.503 | 3.867 | 0.815 | 62.249 | 10.608 | 3.732 | 77.881 | 11.503 | 3.867 |
| 32 | 0.03125 | 4.540 | 3.793 | 1.726 | 92.465 | 4.419 | 3.721 | 112.104 | 4.540 | 3.793 |
| 33 | 0.03030 | 4.398 | 3.717 | 1.689 | 168.771 | 4.279 | 3.679 | 254.258 | 4.398 | 3.717 |
| 34 | 0.02941 | 4.248 | 3.638 | 1.651 | 99.419 | 4.132 | 3.638 | 78.828 | 4.248 | 3.638 |
| 35 | 0.02857 | 2.900 | 3.556 | 1.612 | 86.085 | 2.900 | 3.556 | 241.966 | 2.900 | 3.556 |
| 37 | 0.02703 | 9.079 | 3.563 | 0.559 | 131.707 | 8.905 | 3.348 | 95.728 | 9.079 | 3.563 |
| 38 | 0.02632 | 4.562 | 3.529 | 1.625 | 208.641 | 4.562 | 3.339 | 83.866 | 4.562 | 3.529 |
| 39 | 0.02564 | 4.512 | 3.495 | 1.609 | 171.154 | 4.512 | 3.333 | 124.328 | 4.512 | 3.495 |
| 40 | 0.02500 | 4.460 | 3.461 | 1.592 | 69.468 | 4.460 | 3.312 | 158.228 | 4.460 | 3.461 |
| 41 | 0.02439 | 3.671 | 3.425 | 1.574 | 99.198 | 3.395 | 3.293 | 98.970 | 3.671 | 3.425 |
| 43 | 0.02326 | 10.923 | 3.351 | 0.750 | 126.991 | 10.738 | 3.278 | 186.284 | 10.923 | 3.351 |
| 44 | 0.02273 | 4.217 | 3.312 | 1.520 | 110.338 | 4.217 | 3.262 | 281.353 | 4.217 | 3.312 |
| 45 | 0.02222 | 4.148 | 3.273 | 1.501 | 156.164 | 4.148 | 3.247 | 133.967 | 4.148 | 3.273 |
| 46 | 0.02174 | 4.075 | 3.233 | 1.482 | 89.606 | 4.075 | 3.233 | 122.767 | 4.075 | 3.233 |
| 47 | 0.02128 | 3.999 | 3.192 | 1.462 | 99.771 | 3.999 | 3.192 | 92.319 | 3.999 | 3.192 |
| 48 | 0.02083 | 2.698 | 3.150 | 1.442 | 117.344 | 2.698 | 3.150 | 173.712 | 2.698 | 3.150 |
| Average gap | | 6.468 | 4.453 | 1.507 | 102.828 | 6.146 | 4.190 | 132.336 | 6.024 | 4.344 |

For Procedure 1, let $\lambda_{max}$ be the maximal demand rate for the set of locations. First calculate $\delta = \lambda_{max}(M) - 1/M$. If $\delta < 1e^{-5}$, stop. Otherwise, find the row with maximum demand rate, and then move the facilities above the row toward it by a distance of $\delta/2$.

For Procedure 2, in Step 3.1, we move facility *j* (the one with the smallest demand rate in the neighborhood) toward facility $i_{max}$ by a distance of $d(i_{max},j)(\lambda_{max} - \lambda_j)/2$; in Step 3.2, we move facility $i_{min}$ toward facility *j* (the one with the largest demand rate in the neighborhood) by a distance of $d(i_{min},j)(\lambda_j - \lambda_{min})/2$.

For Procedure 3, in *Step 3*, we move facility $i_{min}$ toward facility $i_{max}$ by a distance of $d(i_{max}, i_{min})(\lambda_{max} - \lambda_{min})/2$.

Table 2 shows the gap between the solutions of the procedures and EFC (i.e., $1/M$). In the table, column "EFC" is the ideal demand rate if an EFC is found, which is equal to $1/M$, column "$H(i,j)$" is the relative gap (in percentage) between the $\lambda_{max}(M)$ found by Procedure *i* started with initial solution set *j* and EFC, which is equal to $(\lambda_{max}(M) - \text{EFC})/\text{EFC}$. Thus, $j = 1$ is an initial solution starting with a random facility location, $j = 2$ is an initial solution based on the solution of the *M-center* problem, and $j = 3$ is based on the rectangle center;

and $i = 0$ stands for the initial solution sets without any adjustment, $i = 1$ stands for the solution using procedure 1 (and therefore is applied only to an initial solution based on the rectangle center), and $i = 2$ and $i = 3$ stand for the solution using Procedures 2 and 3, respectively.

Observe that Table 2 does not include cases where $M = n^2$ and $M = n(n+1)$ because under these case, we can divide the square to $M$ identical rectangles (leading to an EFC). If the coverage threshold $r$ is large, an EFC exists for most such $M$ values. For example, for $M = 9$, 12, and 25 we can divide the unit square into $3 * 3$, $3 * 4$, and $5 * 5$ identical regions respectively.

In this computational experiments, we ignored the coverage constraints. From the table, we can conclude that when the coverage constraints are not very restrictive (our heuristic in next section is based on these conclusions):

- Initial solution set 3, based on the rectangular heuristic, was better than set 2 (using the $M$ – center) that was better then set 1 (random location).
- Procedure 1 was better than Procedures 2 and 3. The performance of procedure 1 was very good. Procedures 2 and 3 did not improve the solution much.

### 4.4. Heuristic for ELP($M$)

Based on the performance of the procedures, we present the following heuristic to generate an approximate solution for ELP($M$) (for feasible $M$ ).

*Step 1*. If $M$ can be represented by the product of two integer numbers, let $m_1$, $m_2$ be the pair that has the smallest difference among all the possible combinations, i.e., $|m_1 - m_2| = \min\{|m_1^i - m_2^i| : M = m_1^i m_2^i\}$. If $(1/m_1)^2 + (1/m_2)^2 \leqslant 4r^2$, then divide $P$ into $m_1$ by $m_2$ rectangles and locate the $M$ facilities at the center of the rectangles. An EFC is reached and $\lambda_{\max}(M) = 1/M$, stop.

*Step 2*. Use the rectangle center method from Section 4.1 to divide $P$ into $M$ rectangles. Locate the facilities at the center of the rectangles. Use Procedure 1 to fine tune the position of the facilities. Check the feasibility (for coverage constraints). If it is feasible, the maximum area is an approximation for $\lambda_{\max}(M)$, stop.

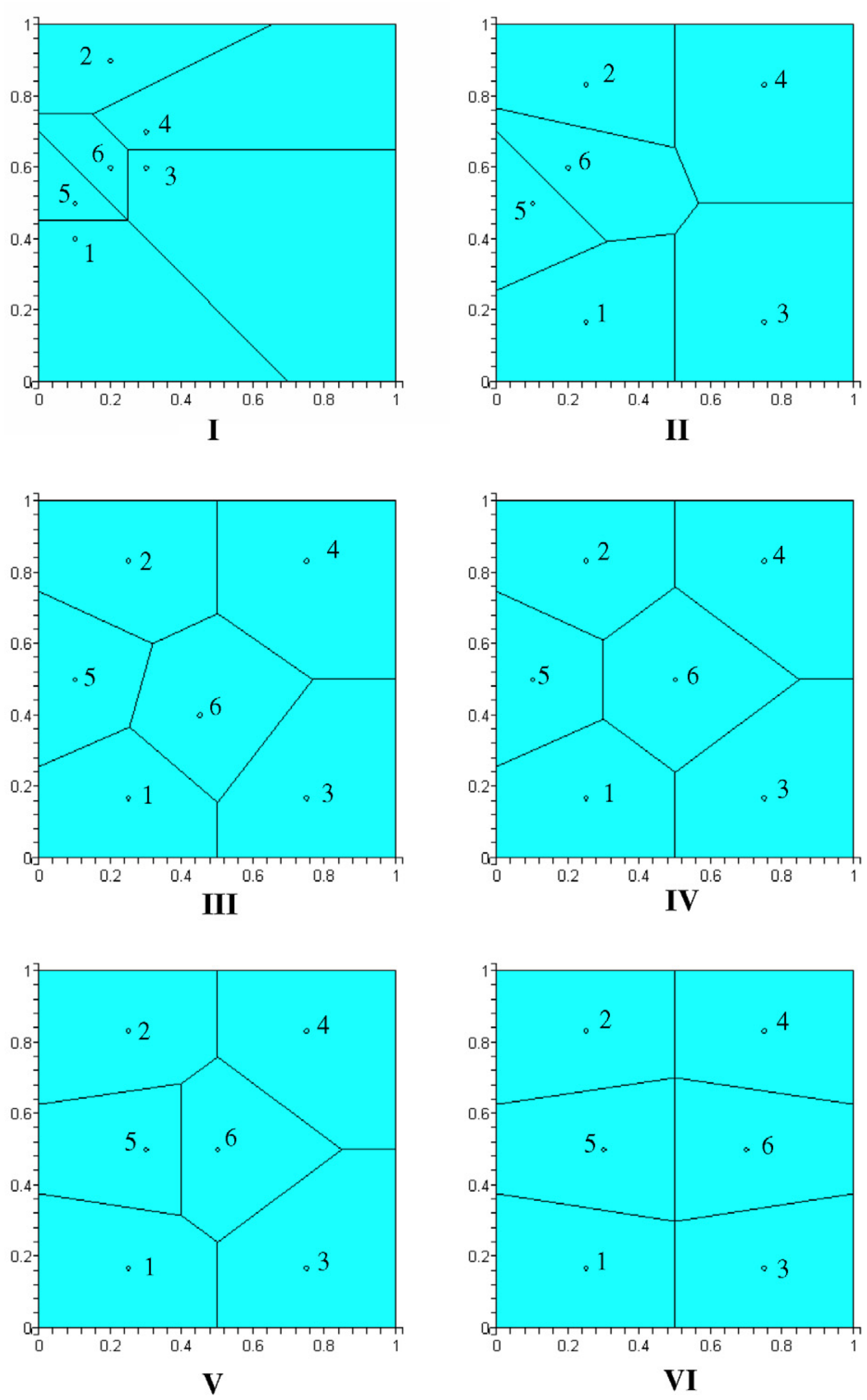
*Step 3*. Find the solution for $M$-center location problem. Execute Procedure 2 or 3 to adjust the locations. The maximum area is an approximation for $\lambda_{\max}(M)$.

## 5. Summary and future work

In this paper, we consider ELP of minimizing the maximal demand rate to a facility, on a plane. Three sets of initial solutions and three procedures to improve these solutions are developed, and a heuristic approach is suggested. Our work focused on the unit plane with uniform demand and indeed many of our results are confined to these settings. However, in view of the available literature on using Voronoi diagrams (e.g., chapter 10 and 12 of Gersho and Gray (1991), Aurenhammer (1991), Suzuki and Okabe (1995), and references therein) we believe that the approach presented, namely formulating ELP using a Voronoi diagram and improving the location of facilities starting from a "good" initial location of facilities, should work in more general settings.

Moreover, as mentioned before, the algorithm of BBK for solving the SCFLP problem depends on the topology of the space only by the solution of ELP. (The other components of their algorithm are completely general.) Therefore, using the results in this paper the SCFLP problem on the unit square with uniform demand can be solved.

## Appendix A. Voronoi diagrams for Example 1



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